JRAHS Year 12 Term 1 Mathematics (2U) 2008

Question 1 **MARKS**

a) Find: i) $\int (x+2)(x-7)\,dx$. **2**

ii) $\int (5-8x)^9\,dx$. **1**

b) Evaluate: i) $\int_0^1 \frac{1}{e^{2x}}\,dx$. **2**

ii) $\int_0^3 \frac{2x-1}{x+4}\,dx$. **3**

c) On the same diagram shade the region where $y < \sqrt{4-x^2}$ and $y \geq x+2$ hold simultaneously. **3**

d) i) Show $\frac{d}{dx}\left(x\sqrt{2x-4}\right) = \frac{x}{\sqrt{2x-4}} + \sqrt{2x-4}$. **1**

ii) Hence find $\int \frac{x}{\sqrt{2x-4}}\,dx$. **3**

Question 2 (Start a new page)

a)

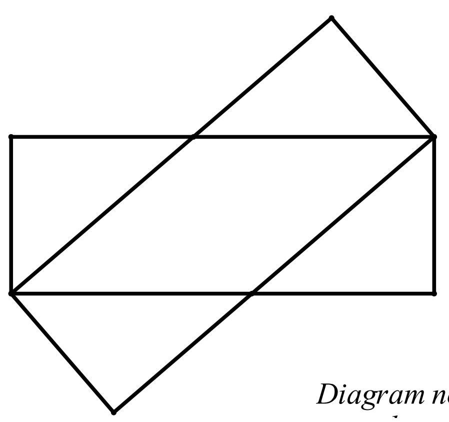


Diagram not

In the above diagram, *ABCD* and *DEBF* are two congruent rectangles with sides 3 and 7 units as in the diagram. ($AB = DF = 7$, $AD = DE = 3$).

Copy the diagram onto your writing booklet.

i) Prove that $\Delta ATD \equiv \Delta FTB$. **2**

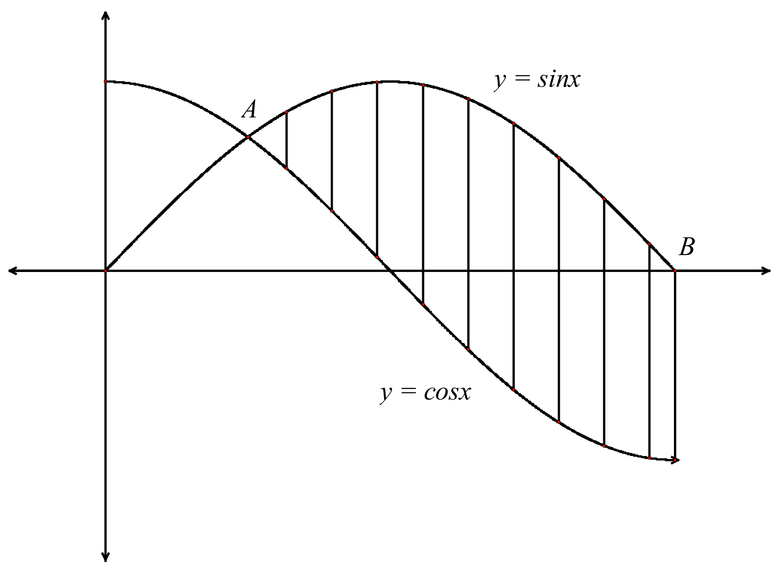
ii) Show that $AT = \frac{20}{7}$. **2**

Question 2 (continued) **MARKS**

b) Without using calculus, sketch $y = \dfrac{x+2}{x-3}$, showing all the essential features. **3**

c) Simplify $\sum_{k=1}^{n} \log\left(\dfrac{k+1}{k}\right)$. **3**

d)



The diagram shows parts of the curves $y = \sin x$ and $y = \cos x$.

i) Find the x-cordinates of the two points A and B. **2**

ii) Calculate the area of the shaded region. **3**

Question 3 (Start a new page)

a) Find $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\,dx$. **2**

b) The area bounded by curve $y = e^{2x}$, the y- axis and the line $y = 3$ is rotated about the y-axis to give a solid.

i) Show that the volume V_y units^3 of the solid formed, is given by **2**

$$V_y = \frac{\pi}{4}\int_1^3 (\ln y)^2\,dy.$$

ii) Use Simpson's rule with 5 function values to find the volume of this solid, correct to 2 decimal places. **3**

Question 3 (continued) **MARKS**

c) Find the area bounded by the graph of $y = x(x-1)(x-2)$ and the x- axis. **3**

d) Consider the geometric series $1 + (3x-2) + (3x-2)^2 + \ldots\ldots$

i) For what values of x does this series have a limiting sum? **3**

ii) Find the value of x if this limiting sum has a value of $\frac{2}{3}$. **2**

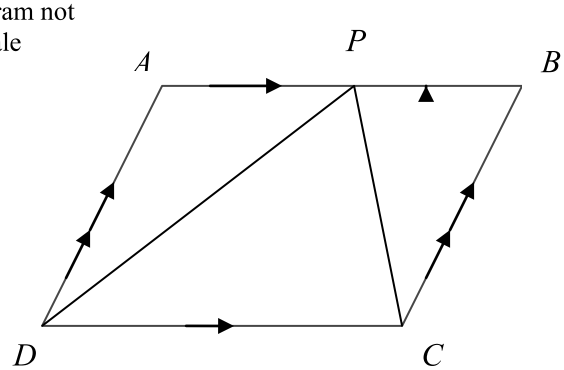
Question 4 (Start a new page)

a) $ABCD$ is a parallelogram and P is on AB such that PD bisects $\angle ADC$ and PC bisects $\angle BCD$.

Copy the diagram onto your writing booklet.

Prove that $AB = 2 \times AD$. **4**

Diagram not to scale



Question 4 (continued)

MARKS

b) With the drought worsening, Ming has designed a counting generator that can simulate the number of rain drops per minute that fall over a pond during a storm.

The rain drops falling per minute forms the sequence:
$\{1, 1, 3, 9, 23, ... \}$
with the nth term given by the formula $R_n = 1 - 2n + 2^n$.

i) Verify that 115 is a term of this sequence. **1**

ii) Find the total number of rain drops which fall over the pond in the first twenty-five minutes. **2**

iii) If the surface area of the pond is 250m^2 find the average number of rain drops per cm^2 over the first twenty five minutes. **1**

c) Katie has planned a holiday which she decides to take in 3 years time. She has estimated that the holiday will cost about \$8000 and plans to save a fixed amount each month for 36 months. She invests her savings at the beginning of each month in an account which pays interest at 6% p.a. compounded monthly at the end of each month.

i) Let the amount she saves each month be $\$A$ and let $\$V_n$ be the value of her investment after n months. Show that the value of her investment at the end of 3 months is given by **1**

$$V_n = A(1.005 + 1.005^2 + 1.005^3)$$

ii) Find correct to the nearest dollar, the least amount of money that Katie would need to save each month to reach her target. **3**

iii) If, after 2 years of her saving plan, the interest rate rose to 9% p.a., how much extra spending money would Katie have if she maintained the amount she was saving as calculated in part(ii) above? **3**

END

JRAHS Yr 12 2u T1 - 08

Q1a i) $\int x^2 - 5x - 14\,dx$ /

$= \frac{x^3}{3} - \frac{5x^2}{2} - 14x + c$ # /

ii) $= \frac{(5-8x)^{10}}{-8 \times 10} + c$

$= \frac{(5-8x)^{10}}{-80} + c$ /

b i) $= \left[\frac{e^{-2x}}{-2}\right]_0^1 = -\frac{1}{2}\left(1 + e^{-2}\right) = \frac{1}{2}(e^{-2} - 1)$ / #

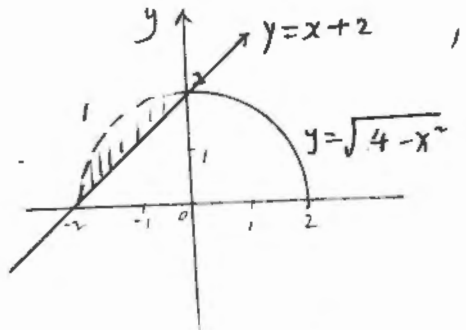
ii) $\int_0^3 \frac{2(x+4)}{x+4} - \frac{9}{x+4}\,dx$

$= \int_0^3 2 - \frac{9}{x+4}\,dx$ /

$= \left[2x - 9\ln(x+4)\right]_0^3$ /

$= 6 - 9\ln 7 - 0 + 9\ln 4$

$= 6 - 9\ln\frac{7}{4}$ / #

c) 

d i) $\frac{d}{dx}\left(x\sqrt{2x-4}\right) = \frac{x \times \cancel{2}}{\cancel{2}\sqrt{2x-4}} + \sqrt{2x-4} \times 1$

$= \frac{x}{\sqrt{2x-4}} + \sqrt{2x-4}$ #

/

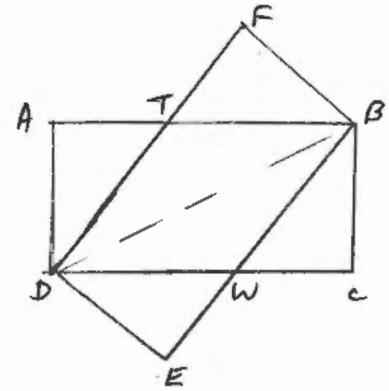
ii) FROM part (i)

$\int \frac{x\,dx}{\sqrt{2x-4}} = x\sqrt{2x-4} - \int \sqrt{2x-4}\,dx$ /

$= x\sqrt{2x-4} - \frac{\cancel{2}}{3}\frac{(2x-4)^{3/2}}{\cancel{2}} + c$ /

$= x\sqrt{2x-4} - \frac{(2x-4)^{3/2}}{3} + c$ /

Q2



i) ABCD and DEBF are both rectangles

$\angle BAD = DFB = 90°$ (angle of rectangle) ½

$\angle ATD = \angle BTF$ (vertically opposite angles) ½

DE = FB (opposite sides of rectangle)

AD = DE = 3 (given)

∴ AD = FB ½

∴ $\Delta ATD \equiv \Delta FTB$ (AAS) ½

or alternatively:

ABCD and DEBF are congruent rectangles

$\Delta ADB \equiv \Delta DFB$ (both equal half the congruent rectangles) /

$\Delta ADB - \Delta TDB \equiv \Delta DFB - \Delta TDB$ /

ie. $\Delta ATD \equiv \Delta FTB$

ii) Let $AT = x$ $TB = 7 - x$

$TF = x$ (corresponding sides of congruent triangles ATD, FTB)

By Pythagoras Th. $(7-x)^2 = x^2 + 3^2$ /

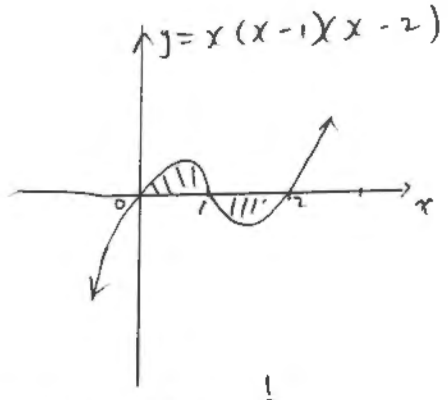
$49 - 14x + \cancel{x^2} = \cancel{x^2} + 9$

$14x = 49 - 9 = 40$

$x = 20/7$ # /

Q 3

c)



Shaded Area $= \int_0^1 x(x-1)(x-2)\,dx +$

$$\left| \int_1^2 x(x-1)(x-2)\,dx \right|$$

$$= \int_0^1 x^3 - 3x^2 + 2x\,dx + \int_2^1 x^3 - 3x^2 + 2x\,dx$$

$$= \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1 + \left[\frac{x^4}{4} - x^3 + x^2 \right]_2^1$$

$$= \left(\tfrac{1}{4} - \cancel{1} + \cancel{1}\right) + \left(\tfrac{1}{4} - \cancel{1} + \cancel{1}\right) - (4 - 8 + 4)$$

$$= \tfrac{1}{4} + \tfrac{1}{4}$$

$$= \tfrac{1}{2} \text{ unit}^2$$

#

d)i)

$s = \frac{1}{1-r}$

$r = 3x - 2$

$|3x-2| < 1$

$-1 < 3x - 2 < 1$

$1 < 3x < 3$

$\frac{1}{3} < x < 1$

but $x \neq \frac{2}{3}$

ii) $\frac{2}{3} = \frac{1}{1-(3x-2)}$

$2 - (6x - 4) = 3$

$6 - 6x = 3$

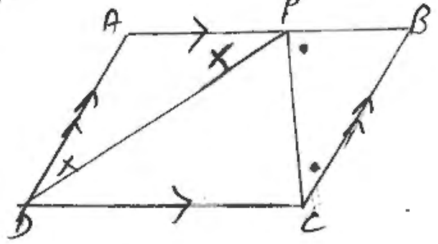
$3 = 6x$

$x = \frac{1}{2}$

#

Q 4

a)



To prove $AB = 2 \times AD$.

Proof :

$AD \parallel BC$ (given)

$\angle APD = \angle PDC$ (alternate angles, $AD \parallel PE$)

But $\angle ADP = \angle PDC$ (PD bisects $\angle ADC$)

$\therefore AD = AP$ (sides opposite equal angles are equal)

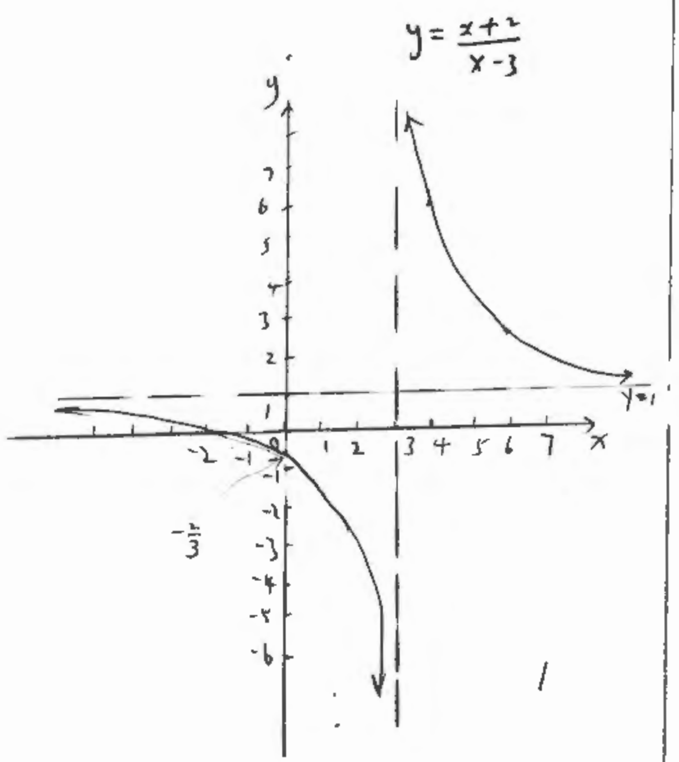
Similarly $BC = BP$

But $AD = BC$ (opposites of parm)

$AD + BC = AP + BP$

$2AD = AB$ *

2b)



/

asymptotes $x = 3$
$y = 1$ /

$x=0, \quad y = -\frac{2}{3}$ /
$y=0, \quad x=-2$

c) $\log\frac{2}{1} + \log\frac{3}{2} + \log\frac{4}{3} + \dots + \log\frac{n+1}{n}$ /

$= \log \frac{\cancel{2}}{1} \times \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{3}} \times \dots \times \frac{\cancel{n}}{\cancel{n-1}} \cdot \frac{n+1}{\cancel{n}}$ /

$= \underline{\underline{\log(n+1)}}$ /

d i) $\sin x = \cos x$ when $x = \frac{\pi}{4}$

$\therefore$ x-coordinate of A $= \frac{\pi}{4}$ # /

$\sin x = 0$ when $x = \pi$

x-coordinate of B $= \pi$ # /

ii) Shaded Area $= \int_{\frac{\pi}{4}}^{\pi} \sin x - \cos x \, dx$ /

$= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\pi}$ /

$= -\left[\cos x + \sin x\right]_{\frac{\pi}{4}}^{\pi}$

$= -(-1+0) + \left(2 \times \frac{1}{\sqrt{2}}\right)$

$= 1 + \frac{2}{\sqrt{2}}$ unit² # /

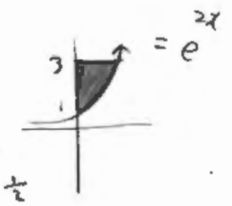
(or $1+\sqrt{2}$ unit²)

Q3

a) $\int e^{\tan x} \sec^2 x \, dx$ /

$= e^{\tan x} + c$ /

b i) $y = e^{2x}$



$\ln y = 2x \quad \therefore \quad x = \frac{1}{2}\ln y$ ½

when $x=0$, $y=1$ ½

Vol (V) $= \int_1^3 \pi x^2 \, dy$ ½

$= \pi \int_1^3 \left(\frac{\ln y}{2}\right)^2 dy$ ½

$= \frac{\pi}{4} \int_1^3 (\ln y)^2 dy$ #

ii)

y	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
$(\ln y)^2$	0	0.1644	0.4805	0.8396	1.2069

/

Vol $= \frac{\pi}{4}\left(\frac{1}{6}\right)\left(0 + 4(0.1644 + 0.8396) + 2\times 0.4805 + 1.2069\right)$ /

$= \frac{\pi}{24}(4.016 + 0.961 + 1.2069)$

$= \underline{\underline{0.81 \ (2dp)}}$ /

Q4

b) $R_n = 1 - 2n + 2^n$

i) $115 = 1 - 2n + 2^n$

When $n = 7$

$1 - 2\cdot 7 + 2^7 = 1 - 14 + 128$ 1

$= 115$

$\therefore$ 115 is a term in the series

ii) $\sum_{n=1}^{25} 1 - 2n + 2^n$

$= 25 - \cancel{2}\left(\frac{1+25}{\cancel{2}}\right)25 + \frac{2(2^{25}-1)}{2-1}$ 1

$= 25 - 26 \times 25 + 2(2^{25}-1)$

$= 67108237$ 1

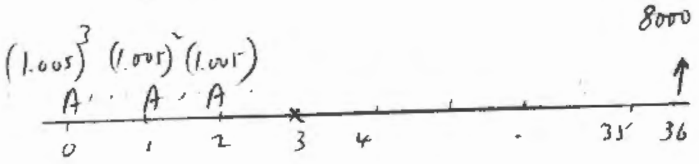
iii) $250\ m^2 = 2{,}500{,}000\ cm^2$

$\therefore$ Ave No of rain drops per cm^2 $= \frac{67108237}{2500000} / cm^2$ 1

$= 26.84 / cm^2$ (2dp) 1

c) 6% p.a $= 0.5\%$ per month

$1 + i = 1.005$



1

$V_3 = A(1.005 + 1.005^2 + 1.005^3)$ #

ii) At the end of 36 months

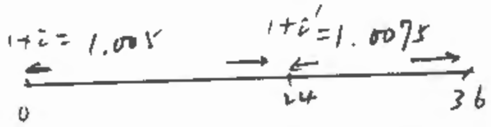
$8000 \leq A(1.005 + 1.005^2 + \cdots + 1.005^{36})$

$8000 \leq A \frac{1.005(1.005^{36}-1)}{1.005-1}$ 1

$A \geq \$202.36$ (2dp) 1

$\therefore$ at least \$203 per month. 1

iii)



Savings at the end of 2 yrs (use A = \$203)

$= 203(1.005)\left(\frac{1.005^{24}-1}{-1+1.005}\right) = 5188.5003$

Balance at the end of 3 yrs

$= 203(1.0075)\left(\frac{1.0075^{12}-1}{1-1.0075}\right) +$ 1

$5188.5003(1.0075)^{12}$

$= 2558.0828 + 5675.2174$ 1

$= 8233.3002$

Extra added value $=$

$8233.3002 - 203 \times 1.005\left(\frac{1.005^{36}-1}{-1+1.005}\right)$

$= 8233.3002 - 8025.1555$

$= \underline{\underline{208.14}}$ (2dp) 1